

CONVEX HULL DEVIATION AND CONTRACTIBILITY

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ABSTRACT. We study the Hausdorff distance between a set and its convex hull. Let X be a Banach space, define the CHD-constant of space X as the supremum of this distance for all subset of the unit ball in X . In the case of finite dimensional Banach spaces we obtain the exact upper bound of the CHD-constant depending on the dimension of the space. We give an upper bound for the CHD-constant in L_p spaces. We prove that CHD-constant is not greater than the maximum of the Lipschitz constants of metric projection operator onto hyperplanes. This implies that for a Hilbert space CHD-constant equals 1. We prove criterion of the Hilbert space and study the contractibility of proximally smooth sets in uniformly convex and uniformly smooth Banach spaces.

1. INTRODUCTION

Let X be a Banach space. For a set $A \subset X$ by ∂A , $\text{int } A$ and $\text{co } A$ we denote the boundary, interior and convex hull of A , respectively. We use $\langle p, x \rangle$ to denote the value of functional $p \in X^*$ at the vector $x \in X$. For $R > 0$ and $c \in X$ we denote by $B_R(c)$ a closed ball with center c and radius R .

By $\rho(x, A)$ we denote distance between the point $x \in X$ and set A . We define the deviation from set A to set B as follows

$$(1) \quad h^+(A, B) = \sup_{x \in A} \rho(x, B).$$

In case $B \subset A$, which takes place below, the deviation $h^+(A, B)$ coincides with the Hausdorff distance between the sets A and B .

Given $D \subset X$ the deviation $h^+(\text{co } D, D)$ is called the *convex hull deviation* (CHD) of D .

We define *CHD-constant* ζ_X of X as

$$\zeta_X = \sup_{D \subset B_1(o)} h^+(\text{co } D, D).$$

Remark 1. Directly from our definition it follows that for any normed linear space X we have $1 \leq \zeta_X \leq 2$.

We denote by ℓ_p^n the n -dimensional real vector space with the p -norm.

This article contains estimates for the CHD-constant for different spaces and some of its geometrical applications. In particular, for finite-dimensional spaces we obtain the exact upper bound of the CHD-constant depending on the dimension of the space:

Theorem 1. *Let X_n be a normed linear space, $\dim X_n = n \geq 2$, then $\zeta_{X_n} \leq 2 \frac{n-1}{n}$. If $X_n = \ell_1^n$ or $X_n = \ell_\infty^n$, then the estimate is reached.*

Let the sets P and Q be the intersections of the unit ball with two parallel affine hyperplanes of dimension k and P is a central section. In Corollary 1 we obtain the exact upper bound of the homothety coefficient, that provides covering of Q by P .

The next theorem gives an estimate for the CHD-constant in the L_p , $1 \leq p \leq +\infty$ spaces:

Theorem 2. *For any $p \in [1, +\infty]$*

$$(2) \quad \zeta_{L_p} \leq 2 \left| \frac{1}{p} - \frac{1}{p'} \right|,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 5 shows that CHD-constant is not greater than the maximum of the Lipschitz constants of metric projection operator onto hyperplanes. This implies that for Hilbert space CHD-constant equals 1. Besides that, we prove the criterion of a Hilbert space in terms of CHD-constant. The idea of the proof is analogous to the idea used by A. L. Garkavi in [1].

Theorem 3. *The equation $\zeta_X = 1$ holds for a Banach space X iff X is an Euclidian space or $\dim X = 2$.*

In addition we study the contractibility of a covering of the convex set with balls.

Definition 1. A covering of a convex set with balls is called *admissible* if it consists of a finite number of balls with centers in this set and the same radii.

Definition 2. A family of balls is called *admissible* when it is an admissible covering of the convex hull of its centers.

We say that a covering of a set by balls is contractible when the union of these balls is contractible. It is easy to show that in two-dimensional and Hilbert spaces any admissible covering is contractible (see Lemmas 2 and 3). On the other hand, using Theorem 3, we prove the following statement.

Theorem 4. *In a three dimensional Banach space X every admissible covering is contractible iff X is a Hilbert space.*

For 3-dimensional spaces we consider an example of an admissible covering of a convex set with four balls that is not contractible. To demonstrate the usefulness of this technics in Theorem 6 we obtain the sufficient condition for the contractibility of the proximally smooth sets in uniformly convex and uniformly smooth Banach space.

2. PROOF OF THEOREM 1 AND SOME OTHER RESULTS

Lemma 1. *Suppose the set $B_1(o) \setminus \text{int } B_r(o_1)$ is nonempty. Then it is arcwise connected.*

Proof.

We suppose that $o \neq o_1$, otherwise the statement is trivial. Let z be the point of intersection of ray o_1o and the boundary of the closed ball $B_1(o)$. The triangle inequality tells us that $B_1(o) \setminus \text{int } B_r(o_1)$ contains z (because the set is nonempty). We claim that $\partial B_1(o) \setminus \text{int } B_r(o_1)$ is arcwise connected and thus prove the lemma. It suffices to show that in the two dimensional case every point of $\partial B_1(o) \setminus \text{int } B_r(o_1)$ is connected with z . Suppose, by contradiction, that it is not true. This means that there exist points $a_1, b_1 \in \partial B_r(o_1) \cap \partial B_1(o)$ lying on the same side of the line oo_1 such that the arc a_1b_1 of the circle $\partial B_1(o)$ contains a point $c \notin B_r(o_1)$, that is $\|c - o_1\| > r$.

Consider two additional rays oa and ob codirectional with o_1a_1 and o_1b_1 respectively, where $a, b \in \partial B_1(o)$. Since balls $B_1(o)$ and $B_r(o_1)$ are similar, we have $a_1b_1 \parallel ab$. So, the

facts that points a, b, a_1, b_1 lie on the same side of oo_1 line, $oa \cap o_1a_1 = \emptyset$, $ob \cap o_1b_1 = \emptyset$ and that a unit ball is convex, imply that segments ab and a_1b_1 lie on the same line, this contradicts $\|c - o_1\| > r$.

Proof of Theorem 1.

Denote $r_n = 2^{\frac{n-1}{n}}$.

Suppose the inequality doesn't hold. It means that there exists a Banach space X_n with dimension $n \geq 2$, a set $D \subset B_1(o) \subset X_n$ and a point $o_1 \in \text{co } D$, such that $B_{r_n}(o_1) \cap D = \emptyset$. But if $o_1 \in \text{co } D$, then $o_1 \in \text{co}(B_1(o) \setminus \text{int } B_{r_n}(o_1))$. According to Lemma 1 the set $B = B_1(o) \setminus \text{int } B_{r_n}(o_1)$ is connected. So, taking into consideration the generalized Caratheodory's theorem ([2], theorem 2.29), we see that the point o_1 is a convex combination of not more than n points from B . These points denoted as a_1, \dots, a_k , $k \leq n$, may be regarded as vertices of a $(k-1)$ -dimensional simplex A and point $o_1 = \alpha_1 a_1 + \dots + \alpha_k a_k$ lies in its relative interior ($\alpha_i > 0, \alpha_1 + \dots + \alpha_k = 1$).

Let c_l be the point of intersection of ray $a_l o_1$ with the opposite facet of the simplex A . So, $o_1 = \alpha_l a_l + (1 - \alpha_l) c_l$. Then

$$\|o_1 - a_l\| = (1 - \alpha_l) \|c_l - a_l\|.$$

And $[c_l, a_l] \subset A \subset B_1(o)$ implies that $\|a_l - c_l\| \leq 2$, for all $l \in \overline{1, k}$. Therefore $r_n < \|o_1 - a_l\| \leq 2(1 - \alpha_l)$. Thus $\alpha_l < 1 - \frac{r_n}{2} < \frac{1}{n}$, and finally $\alpha_1 + \dots + \alpha_k < \frac{k}{n} \leq 1$. Contradiction.

Now let us show that the estimate is attained for spaces ℓ_1^n, ℓ_∞^n .

Consider ℓ_1^n . Let $A = \{e_i\}_{i=1}^n$ be a standard basis for ℓ_1^n space and $b = \frac{1}{n}(e_1 + \dots + e_n) \in \text{co}\{e_1, \dots, e_n\}$. The distance between point b and an arbitrary point from A is $\|a_i - b\| = 2^{\frac{n-1}{n}}$.

Consider ℓ_∞^n . Let $a_{ij} = (-1)^{\delta_{ij}}$, where δ_{ij} is Kroneker symbol, $a_i = (a_{i1}, \dots, a_{in})$ and $A = \{a_i\}_{i=1}^n$. Now let $b = \frac{1}{n}(a_1 + \dots + a_n) = (\frac{n-2}{n}, \dots, \frac{n-2}{n}) \in \text{co}\{a_1, \dots, a_n\}$. And the distance from point b to an arbitrary point from A is $\|a_i - b\| = 2^{\frac{n-1}{n}}$. ■

So, Theorem 1 and inequality $\zeta_X \geq 1$ imply the CHD-constant of any 2-dimensional normed space equals 1. Obviously, CHD-constant of ℓ_1 space equals 2.

Remark 2. Let X be a Banach space, $\dim X = n$. Then for every $d < \zeta_X$ there exists a set A that consists of not more than n points and meets the condition $h^+(\text{co } A, A) = d$.

Corollary 1. Let sets P and Q be plane sections of the unit ball with two parallel hyperplanes of dimension k , and let the hyperplane containing P contains 0 as well. Then it is possible to cover Q with the set $\min\{2^{\frac{k}{k+1}}; \zeta_X\}P$ using parallel translation.

Proof.

Define $\eta = \min\{2^{\frac{k}{k+1}}; \zeta_X\}$. Due Helly theorem it suffices to prove that we could cover any k -simplex $\Delta \subset Q$ with the set ηP .

Let us consider k -simplex $\Delta \subset Q$ with vertexes $\{x_1, \dots, x_{k+1}\}$. Due to the definition of the ζ_X and by Theorem 1 for any set of indices $I \subset \overline{1, (k+1)}$, we have $\text{co}\{x_i\}_{i \in I} \subset \bigcup_{i \in I} (B_\eta(x_i) \cap \Delta)$. Using KKM theorem [14] we obtain that $S = \bigcap_{i \in \overline{1, (k+1)}} (B_\eta(x_i) \cap \Delta) \neq \emptyset$. Then $\Delta \subset B_\eta(s)$, where $s \in S \subset \Delta$. ■

Let us show that Hilbert and 2-dimensional Banach spaces meet the requirements of Theorem 4. We consider the area covered with balls to be shaded. Balls' radii may be taken equal to 1.

Lemma 2. *Let X be a Banach space, $\dim X = 2$, then any admissible covering is contractible.*

Proof.

Without loss of generality, let we have an admissible covering of a convex set V by balls $B_1(a_i)$, $i = \overline{1, n}$. Let us put $S = \bigcup_{i \in \overline{1, n}} B_1(a_i)$. Since the unit ball is a convex

closed body, the set S is homotopy equivalent to its nerve [4], in our case it is finite CW complex. Therefore, S is contractible iff S is connected, simply connected and its homology groups $H_k(S)$ are trivial for $k \geq 2$. Obviously, S is connected set.

Let us show that the set S is simply connected and $H_k(S) = 0$ for $k \geq 2$. The unit circle is a continuous closed line without self-intersections, it divides a plane in two parts. A finite set of circles divides a plane in a finite number of connected components. Let us now shade the unit balls.

It is remarkable, that the problem is stable against subtle perturbations of norm. To be more precise: if a norm does not meet the requirements of the theorem, then there exists a polygon norm, which does not meet them too.

Let us choose a bounded not-covered area U with shaded boundary. It is possible to put a ball of radius $3\varepsilon_1$ ($\varepsilon_1 > 0$) inside this area. There exists ε_2 ($\varepsilon_2 > 0$) such that if $B_1(a_{i_1}) \cap B_1(a_{i_2}) = \emptyset$ for $i_1, i_2 \in \overline{1, n}$, than $B_{1+\varepsilon_2}(a_{i_1}) \cap B_{1+\varepsilon_2}(a_{i_2}) = \emptyset$. Denote $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

Consider the following set

$$B_1^c(o) = \bigcap_{p \in C} \{x : \langle p, x \rangle \leq 1\},$$

where C is a finite set of unit vectors from space X^* , such that $C = -C$. So, $B_1^c(o)$ is the unit ball for some norm. According to [5], Corollary 2.6.1, it is possible to pick such a set C , that $h^+(B_1^c(o), B_1(o)) \leq \varepsilon$. Then the set of balls $B_1^c(a_i)$, $i = \overline{1, n}$ is admissible covering, contains the boundary of U , because $B_1(o) \subset B_1^c(o)$, and it does not cover U entirely. Furthermore nerve, and consequently homology group, of the sets $\bigcup_{i \in \overline{1, n}} B_1^c(a_i)$

and S are coincide.

Now it suffices to show that the statement of the lemma is true in case of a polygon norm. In this case S is the neighborhood retract in \mathbb{R}^2 (see [6]), therefore straightforward from Alexander duality (see [7], Chapter 4, §6) we obtain that $H_k(S) = 0$ for $k \geq 2$.

Now we shall prove that S is simply connected. Assume the contrary, there exist a norm, an admissible covering of a convex set V by balls $B_1(a_i)$, $i = \overline{1, n}$ and non-shaded bounded set U with a shaded boundary. Note that its boundary appears to be a closed polygonal line without self-intersections. Let us define set $A = \text{co}\{a_i | i = \overline{1, n}\}$.

Let x be an arbitrary point of the set U . The union of the balls $B_1(a_i)$ is admissible covering of the set A , thus $x \notin A$. Then there exists a line l_a that separates x from set A . This line may serve as a supporting line of set A . Let $l \parallel l_a$ be a supporting line of U in a point v , such that sets U and A lie at one side from line l . Line l divides the plane in two semiplanes. Let H_+ be the semiplane that does not contain A , we denote

the other semiplane as H_- . Let points $p, q \in l$ lie on different sides from v . We want to choose all the edges of polygonal curve ∂U , that contain point v . We will call them $vb_i, i \in \overline{1, k} : \cos \angle p v b_i > \cos \angle p v b_j, i > j$.

Note that it is impossible for any of the edges to lie on line l . Otherwise l is supporting line for a ball $B_1(a_p), p \in \overline{1, n}$, and $B_1(a_p) \cap H_+ \neq \emptyset$, so we come to the contradiction. We may pick such a number ε that the ball $B_\varepsilon(v)$ intersects only with particular edges of polygonal curve ∂U . From now on we use $p, q, b_i, i \in \overline{1, k}$ for points of intersection of circle $\partial B_\varepsilon(v)$ with corresponding edges. Since $v \in \partial U$, it follows that there exists a point z on circle $\partial B_\varepsilon(v)$, such that the interior of segment vz lies in U and the ray vz lies between vb_1 and vb_k . Then, since the ball is convex, there is no such ball $B_1(a_i)$, that simultaneously covers a point from the interior of vb_1 and a point from vb_k , i.e. point v is covered by at least two balls, and the centers of these balls a_i, a_j are divided by ray vz in semiplane H_- . Again, since the ball is convex, point $x = vz \cap a_i a_j$ is not covered by balls $B_1(a_i), B_1(a_j)$, thus $\|a_i - a_j\| = \|x - a_i\| + \|x - a_j\| > 2$, which contradicts the fact that a_i and a_j are contained in ball $B_1(v)$. ■

Lemma 3. *Let X be an Euclidean space. Then any admissible covering is contractible.*

Proof.

Let us remind that a closed convex set is contractible and in a Hilbert space the projection onto a closed convex set is unique. Since a projection onto a convex set is a continuous function of the projected point, it is enough to prove that a line segment, which connects a shaded point with its projection onto a convex hull of centers of an admissible covering, is shaded. Suppose that we have an admissible set of balls. The convex hull of its center is a polygon, Let us call it C . If a shaded point a is projected onto the v -vertex of the polygon, then the segment av is shaded as well. Let a shaded point a lying in the ball $B_1(v)$ from a set of balls be projected onto the point $b \neq v$. Let L be a hyperplane passing through point b and perpendicular to the line segment $[a, b]$. It divides the space in two half-spaces. The one with the point a we call H_a . C is convex, thus it contains the segment $[v, b]$. Then it is impossible for point v to lie in H_a , so $\angle abv \geq \frac{\pi}{2}$, i.e. $\|v - a\| \geq \|v - b\|$. Thus, $b \in B_1(v)$ and, consequently, $ab \subset B_1(v)$. ■

3. UPPER BOUND FOR CHD-CONSTANT IN A BANACH SPACE

Let $J_1(x) = \{p \in X^* \mid \langle p, x \rangle = \|p\| \cdot \|x\| = \|x\|\}$. Let us introduce the following characteristic of a space:

$$\xi_X = \sup_{\substack{\|x\|=1, \\ \|y\|=1}} \sup_{p \in J_1(y)} \|x - \langle p, x \rangle y\|,$$

Note that if $y \in \partial B_1(0)$, $p \in J_1(y)$, then vector $(x - \langle p, x \rangle y)$ is a metric projection of x onto the hyperplane $H_p = \{x \in X : \langle p, x \rangle = 0\}$. So, $\xi_X = \sup_{y \in B_1(0)} \sup_{p \in J_1(y)} \xi_X^p$, where ξ_X^p is half of diameter of a unit ball's projection onto the hyperplane H_p . This implies the following remark.

Let us use ξ_X for estimation of CHD-constant of X :

Lemma 4. *Let $y \in \text{co}[B_1(0) \setminus \text{int } B_r(y_1)]$ and let $p \in J_1(y)$. There is $x \in B_1(0) \setminus \text{int } B_r(y_1)$ such that $\langle p, x \rangle = \langle p, y \rangle$.*

Then in hyperplane $H_p = \{x \in X : \langle p, x \rangle = \langle p, o_1 \rangle\}$ there exists a point x , such that $x \in B_1(o) \setminus \text{int } B_r(o_1)$.

Proof.

Define set $B = B_1(o) \setminus \text{int } B_r(y)$. Since $y \in \text{co } B$, there exist points $a_1, \dots, a_n \in B$ and a set of positive coefficients $\lambda_1, \dots, \lambda_n$ ($\lambda_1 + \dots + \lambda_n = 1$), such that

$$(3) \quad y = \lambda_1 a_1 + \dots + \lambda_n a_n.$$

Let $H_p^+ = \{x \in X : \langle p, x \rangle \geq \langle p, y \rangle\}$. According to Lemma 1 set B is connected, thus, since $B \setminus H_p^+$ is not empty, if the statement we prove is not true, we arrive at $B \cap H_p^+ = \emptyset$. Then $\langle p, a_i \rangle < \langle p, y \rangle$ and formula (3) implies

$$\langle p, y \rangle = \lambda_1 \langle p, a_1 \rangle + \dots + \lambda_n \langle p, a_n \rangle < \langle p, y \rangle.$$

Contradiction. ■

Lemma 5.

$$(4) \quad \zeta_X \leq \sup_{\|y\|=1} \inf_{p \in J_1(y)} \sup_{\substack{x \in B_1(o): \\ \langle p, x-y \rangle = 0}} \|x - y\|$$

Proof.

Let ε be a positive real number. Then, according to the definition of the CHD-constant, there exists set $D \subset B_1(o)$, such that $h^+(\text{co } D, D) \geq \zeta_X - \varepsilon$. It means that there exists point $y \in \text{co } D : \rho(y, D) \geq \zeta_X - 2\varepsilon$. Let us put $r = \rho(y, D)$. So, $D \subset B_1(o) \setminus \text{int } B_r(y)$. Hence, $y \in \text{co } [B_1(o) \setminus \text{int } B_r(y)]$. Now let $p \in J_1(y)$.

According to Lemma 4 there exists vector $x \in B_1(o) \setminus \text{int } B_r(y) : \langle p, x - y \rangle = 0$. And $r \leq \|x - y\|$. Therefore, $\zeta_X \leq \rho(y, D) + 2\varepsilon = r + 2\varepsilon \leq \|x - y\| + 2\varepsilon$. Now let ε tend to zero. The lemma is proved. ■

It becomes obvious that

$$\xi_X = \sup_{\substack{y \in B_1(o), \\ p \in J_1(y)}} \sup_{\substack{x \in B_1(o): \\ \langle p, x-y \rangle = 0}} \|x - y\|.$$

Then Lemma 5 implies

Theorem 5. $\zeta_X \leq \xi_X$.

Using Remark 1 and Theorem 5 we get

Corollary 2. If H is a Hilbert space, then $\zeta_H = 1$.

With the following lemma we can pass to finite subspace limit in CHD-constant calculations.

Lemma 6. Let X be a Banach space and $\{x_1, x_2, \dots\}$ be a vector system in it, such that the subspace $\tilde{X} = \text{Lin}\{x_1, x_2, \dots\}$ is dense in X . Then

$$(5) \quad \zeta_X = \lim_{n \rightarrow \infty} \zeta_{X_n},$$

where $X_n = \text{Lin}\{x_1, \dots, x_n\}$.

Proof.

Let us set $\zeta = \zeta_X$, and fix a real number $\varepsilon > 0$. Since $X_n \subset X_{n+1} \subset X$, the sequence ζ_{X_n} is monotone and bounded and, consequently, convergent. Let $\zeta_2 = \lim_{n \rightarrow \infty} \zeta_{X_n}$. Since

$X_n \subset X$ it follows that $\zeta_2 \leq \zeta$. According to the CHD-constant definition there exists a set $A \subset B_1(o)$ and a point $d \in \text{co } A$, such that $\rho(d, A) > \zeta - \frac{\varepsilon}{2}$. Since $d \in \text{co } A$, there exist a natural number N , points $a_i \in A$, and numbers $\alpha_i \geq 0$, $i \in \overline{1, N}$, $\alpha_1 + \dots + \alpha_N = 1$, such that $d = \alpha_1 a_1 + \dots + \alpha_N a_N$.

Then $\|d - a_i\| > \zeta - \frac{\varepsilon}{2}$, $i \in \overline{1, N}$. Since $\overline{X} = X$, it is possible to pick points $b_i \in B_1(o) \cap \check{X}$, $i \in \overline{1, N}$, so that $\|a_i - b_i\| \leq \frac{\varepsilon}{4}$. According to the definition of a linear span for some natural n_i we have: $b_i \in X_{n_i}$. Let $M = \max n_i, i \in \overline{1, N}$. Consider set $B = \{b_1, \dots, b_N\}$ in the space X_M . Let $d_\varepsilon = \alpha_1 b_1 + \dots + \alpha_N b_N \in \text{co } B$, then

$$\|d_\varepsilon - d\| = \left\| \sum_{j=1}^N \alpha_j (b_j - a_j) \right\| \leq \sum_{j=1}^N \alpha_j \|b_j - a_j\| \leq \frac{\varepsilon}{4},$$

so for every $i \in \overline{1, N}$ we have

$$\|d_\varepsilon - b_i\| = \|(d_\varepsilon - d) + (d - a_i) + (a_i - b_i)\| \geq \|d - a_i\| - \|d_\varepsilon - d\| - \|a_i - b_i\| \geq \zeta - \varepsilon.$$

Thus $\zeta - \varepsilon \leq h^+(\text{co } B, B) \leq \zeta_{X_M} \leq \zeta_2 \leq \zeta$, and since $\varepsilon > 0$ was chosen arbitrarily, $\zeta = \zeta_2$. ■

Let $p' \in [1; +\infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$, $r = \min\{p, p'\}$, $r' = \max\{p, p'\}$.

Lemma 7. *Given $p \in [1, +\infty]$. Let $x_i \in L_p, 1 \leq p \leq \infty, i = 1, \dots, k$;*

$$\sum_{i=1}^k \alpha_i = 1, \alpha_i \geq 0 \ (i = 1, \dots, k), \ x_0 = \sum_{i=1}^k \alpha_i x_i.$$

Then

$$(6) \quad \left(\sum_{i=1}^k \alpha_i \|x_i - x_0\|_p^r \right)^{\frac{1}{r}} \leq 2^{-\frac{1}{r'}} \left(\sum_{i=1, j=1}^k \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{\frac{1}{r}},$$

$$(7) \quad \left(\sum_{i=1, j=1}^k \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{\frac{1}{r}} \leq 2^{\frac{1}{r}} \max_{1 \leq i \leq k} \|x_i\|_p.$$

If $1 \leq p \leq 2$, then the latter inequality can be strengthened:

$$(8) \quad \left(\sum_{i=1, j=1}^k \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{\frac{1}{r}} \leq 2^{\frac{1}{r}} \left(\frac{k-1}{k} \right)^{\frac{2}{p}-1} \max_{1 \leq i \leq k} \|x_i\|_p.$$

Proof.

The inequality (7) follow from Schoenberg's inequalities ([8], Theorem 15.1):

$$\left(\sum_{i=1, j=1}^k \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{\frac{1}{r}} \leq 2^{\frac{1}{r}} \left(\max_{1 \leq i \leq k} \{1 - \alpha_i\} \right)^{\frac{2}{r}-1} \left(\sum_{i=1}^k \alpha_i \|x_i\|_p^r \right)^{\frac{1}{r}}.$$

The inequality (8) was deduced by S.A. Pichugov and V.I. Ivanov in ([9], Assertion 1).

Using the Riesz-Thorin theorem for spaces with a mixed L_p -norm ([8], §14), S.A. Pichugov proved the following inequality ([10], Theorem 1):

$$(9) \quad \left(\sum_{i=1}^k \sum_{j=1}^l \alpha_i \beta_j \| (x_i - x_0) - (y_j - y_0) \|_p^r \right)^{\frac{1}{r}} \leq 2^{-\frac{1}{r'}} \left(\sum_{i_1=1, i_2=1}^k \alpha_{i_1} \alpha_{i_2} \| x_{i_1} - x_{i_2} \|_p^r + \sum_{j_1=1, j_2=1}^l \beta_{j_1} \beta_{j_2} \| y_{j_1} - y_{j_2} \|_p^r \right)^{\frac{1}{r}},$$

where $\sum_{i=1}^k \alpha_i = \sum_{j=1}^l \beta_j = 1$, $\alpha_i \geq 0$ ($i = 1, \dots, k$), $\beta_j \geq 0$ ($j = 1, \dots, l$), $x_0 = \sum_{i=1}^k \alpha_i x_i$, $y_0 = \sum_{j=1}^l \beta_j y_j$.

Substituting y_j for 0 and β_j for $\frac{1}{l}$ in (9) we obtain the inequality (6). ■

Proof of Theorem 2.

Consider the case of $p \in (1, +\infty)$.

For L_p spaces and arbitrary set of vectors $A = \{x_0, x_1, \dots, x_k\}$, such that $x_0 = \sum_{i=1}^k \alpha_i x_i$, $\sum_{i=1}^k \alpha_i = 1$, $\alpha_i \geq 0$ ($i \in \overline{1, k}$), $A \subset B_1(0)$ we have

$$\left(\min_{i \in \overline{1, k}} \|x_0 - x_i\|_p^r \right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^k \alpha_i \|x_i - x_0\|_p^r \right)^{\frac{1}{r}}.$$

Using (6) and (7), since set of vectors A was chosen arbitrarily, we get $\zeta_{L_p} \leq 2^{(\frac{1}{r} - \frac{1}{r'})} = 2^{|\frac{1}{p} - \frac{1}{p'}|}$.

As it was shown in proof of Theorem 1 that $\zeta_{\ell_1^n} = \zeta_{\ell_\infty^n} = 2^{\frac{n-1}{n}}$. Thus, $\zeta_{L_1} = \zeta_{L_\infty} = 2$. ■

Remark 3. If $1 \leq p \leq 2$, then, using in the proof of Theorem 2 inequality (8) instead of (7), we arrive at:

$$(10) \quad \zeta_{\ell_p^n} \leq \left(2 \frac{n-1}{n} \right)^{|\frac{1}{p} - \frac{1}{p'}|}.$$

Still without any answer remains the following questions:

Question 1. Is the inequality (10) true if $p \in (2; \infty)$?

Question 2. Is the estimate in the inequality (2) exact in case of $p \in (1; \infty)$, $p \neq 2$?

4. CRITERION OF A HILBERT SPACE

In order to prove Theorem 3 we need the following lemma, which follows directly from the KKM theorem [14].

Lemma 8. Let X be a Banach space. Suppose the triangle $a_1 a_2 a_3 \subset X$ satisfies the inequality $\text{diam } a_1 a_2 a_3 \leq 2R$ and is covered by balls $B_R(a_i)$, $i = 1, 2, 3$. Then these balls have a common point lying in the plane of the triangle.

Taking into account Lemma 8, the proof of Theorem 3 is very similar to the one of Theorem 5 from [1].

Proof of theorem 3.

Using Theorem 1 and Corollary 2 it suffices to prove that a Banach space X , with $\dim X \geq 3$ and $\zeta_X = 1$ is a Hilbert space. According to the well-known results obtained by Frechet and Blashke-Kakutani, it is enough to describe only the case when $\dim X = 3$. We need to show that if $\zeta_X = 1$, then for every 2-dimensional subspace there exists a unit-norm operator that projects X onto this particular subspace. Let $0 \in L$ be an arbitrary 2-dimensional subspace in X , point c is not contained in L . We denote $B_n^2(0) = L \cap B_n(0)$ (it is a ball of radius $n \in \mathbb{N}$ in space L). For every $n \in \mathbb{N}$ let us introduce the following notations:

$$\begin{aligned} E_n &= \{x \in L : \|c - x\| \leq n\}, \\ F_n &= \{x \in L : \|c - x\| = n\}. \end{aligned}$$

If n is big enough, these sets are nonempty. Let x_1, x_2, x_3 be arbitrary points from E_n . The CHD-constant of space X equals 1, so the balls $B_n^2(x_i), i = 1, 2, 3$ cover the triangle $x_1x_2x_3$. According to Lemma 8, their intersection is not empty. According to Helly theorem, the set

$$S_n = \bigcap_{x \in E_n} B_n^2(x)$$

is nonempty as well.

Let us pick $a_n \in S_n$, then by construction we have

$$(11) \quad \|x - a_n\| \leq \|x - c\|$$

for every $x \in F_n$. Let us show that

$$\|x - a_n\| \leq \|x - c\|$$

for every $x \in E_n$. Suppose that for some $x \in E_n$

$$(12) \quad \|x - a_n\| > \|x - c\|.$$

According to (11) we may assume that $x \in E_n \setminus F_n$. Set E_n is bounded and its boundary relatively to subspace L coincides with F_n , thus there exists point $b \in F_n$, such that x is contained in interval (a_n, b) . Then $a_n - x = \lambda(a_n - b)$, $0 < \lambda < 1$.

Note that $c - x = (c - a_n) + (a_n - x) = c - a_n + \lambda(a_n - b)$, then (12) may be reformulated as $\|c - a_n + \lambda(a_n - b)\| < \lambda \|a_n - b\|$.

So,

$$\begin{aligned} \|c - b\| &= \|(c - a_n) + \lambda(a_n - b) + (1 - \lambda)(a_n - b)\| \leq \\ &\leq \|(c - a_n) + \lambda(a_n - b)\| + (1 - \lambda) \|a_n - b\| < \lambda \|a_n - b\| + (1 - \lambda) \|a_n - b\| = \|a_n - b\|, \end{aligned}$$

and it contradicts (11).

Consider the sequence $\{a_n\}$. Note that $E_n \subset E_{n+1}$ and $\bigcup_{i=1}^{\infty} E_i = L$. So, starting with a fixed natural k , the inclusion $0 \in E_n$, $n \geq k$ becomes true, thus when $x = 0$ inequality (11) implies $\|a_n\| \leq \|c\|$, $n \geq k$, i.e. the sequence $\{a_n\}$ is bounded. It means that sequence $\{a_n\}$ has a limit point a . Then every point $x \in L$ satisfies $\|x - a\| \leq \|x - c\|$. Let now represent every element $z \in X$ in the form

$$z = tc + x \quad (x \in L, t \in \mathbb{R}).$$

Operator $P(z) = P(tc + x) = ta + x$ projects X onto L .

In addition:

$$\|P(z)\| = \|ta + x\| = |t| \left\| a + \frac{x}{t} \right\| \leq |t| \left\| c + \frac{x}{t} \right\| = \|tc + x\| = \|z\|.$$

Hence, the $\|P\| = 1$ and taking into consideration the theorem of Blaschke and Kakutani we come to a conclusion that X is a Hilbert space. ■

Proof of Theorem 4.

It remains to check that in every Banach space X that is not a Hilbert one, where $\dim X = 3$, there exist a convex set and an admissible and not contractible covering.

To make the proof easier we first need to prove a trivial statement from geometry.

Let hyperplane H divide space X in two half-spaces H_+, H_- . Let M be a bounded set in H . We want to cover set M with balls $B = \{\cup B_d(a_i) \mid i \in \overline{1, n}, n \in \mathbb{N}\}$ and call this covering (ε, r, H_+) -good if $h^+(B, H_-) \leq \varepsilon$.

Lemma 9. *Let X be a Banach space, $3 \leq \dim X < +\infty$. Let hyperplane H divide X in two half-spaces: H_+ and H_- . Let M be a bounded set in H . Then for every $\varepsilon > 0$, $d > 0$ there exists an admissible set of balls $B_d(a_i), i \in \overline{1, N}, N \in \mathbb{N}$, such that set $B = \bigcup_{i \in \overline{1, N}} B_d(a_i)$ may be regarded as (ε, d, H_+) -good covering of set M and $\text{co}(M \cup \{a_i\}) \subset B, i \in \overline{1, N}$.*

Proof.

Let $\dim X = n$. Without loss of generality we assume that $\varepsilon < d$ and H is the supporting hyperplane for the ball $B_d(0)$ and $B_d(0) \subset H_-$. For any $r > 0$ and $a \in X$ we use $C_r(a)$ to denote a $(n - 1)$ -dimensional hypercube centered in a that lies in the hyperplane parallel to H , where r is the length of its edges. Let $x \in H \cap B_d(0)$. Then $h^+(B_d(\varepsilon \frac{x}{\|x\|}), H_-) \leq \varepsilon$. Let $D = B_d(\varepsilon \frac{x}{\|x\|}) \cap H$. Note that x is an inner point of set D relatively to subspace L . In a finite dimensional linear space all norms are equivalent, so $C_r(x) \subset D$ for some $r > 0$. As the ball $B_d(\varepsilon \frac{x}{\|x\|})$ is centrally-symmetric, it contains affine hypercube $\text{co}(C_r(x) \cup C_r(\varepsilon \frac{x}{\|x\|}))$. Consider next an arbitrary bounded set $M \subset L$. Since it is bounded, $M \subset C_R(b)$, where $b \in L, R > 0$. We suppose that $R = kr, k \in \mathbb{N}$. Let's split hypercube $C_R(b)$ in hypercubes with edges of length r and let $b_i, i \in \overline{1, N}$ be the centers of these hypercubes. Hence, from the above, the balls $B_d(b_i - (d - \varepsilon) \frac{x}{\|x\|})$ give us the necessary covering. ■

Let us consider an approach to construct an admissible and not contractible covering of a convex set.

Let a Banach space X be a non-Hilbert one and $\dim X = 3$. According to Theorem 3, $\zeta_X > 1$, by Remark 2, there exists set $A = \{a_1, a_2, a_3\} \subset B_1(0)$ and point $b \in \text{co } A$, such that $\rho(b, A) = 1 + 4\varepsilon > 1$. According to Theorem 1, $o \notin H$. Consider the balls $B_{1+\varepsilon}(a_i), i \in \overline{1, 3}$, let $B_1 = B_{1+\varepsilon}(a_1) \cup B_{1+\varepsilon}(a_2) \cup B_{1+\varepsilon}(a_3)$. It is obvious that $b \notin B_1$. Since all the edges of triangle $a_1a_2a_3$ lie in B_1 , facets $0a_1a_2, 0a_1a_3, 0a_2a_3$ of tetrahedron $0a_1a_2a_3$ lie in B_1 . Let H be a plane passing through points a_1, a_2, a_3 .

Let H divide space X in two half-spaces: H_+ and H_- . Let $0 \in H_+$. According to Lemma 9 there exists an $(\varepsilon, 1 + \varepsilon, H_+)$ -good covering of triangle $a_1a_2a_3$ with an admissible set of balls that have centers lying in a set $C = \{c_i, i \in \overline{1, N}\}, N \in \mathbb{N}$. Let

$B_2 = \bigcup_{i \in \overline{1, N}} B_{1+\varepsilon}(c_i)$. Then set $B = B_1 \cup B_2$ contains all the facets of tetrahedron $oa_1a_2a_3$

and does not contain interior of ball $B_\varepsilon\left(b - 2\varepsilon \frac{b}{\|b\|}\right)$, i.e. set B is not contractible. However, $\text{co}(A \cup C) \subset B_2 \subset B$, i.e. union of balls $B_{1+\varepsilon}(x)$, $x \in A \cup C$ is an admissible covering for the set $\text{co}(A \cup C)$ we were looking for. ■

There remain still some open questions:

Question 3. *What is the minimal number of balls in an admissible and not contractible set of balls for a certain space X ? How to express this number in terms of space characteristics, such as its dimension, modulus of smoothness and modulus of convexity?*

Question 4. *How to estimate the minimal density of an admissible covering with balls for it to be contractible?*

According to Lemma 8, it takes at least 4 balls to construct an admissible not contractible set of balls in an arbitrary Banach space. The following example describes the case with precisely 4 balls.

Example 1. *Let $X = l_1^3$, $a_1 = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$, $a_2 = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$, $a_3 = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$, $a_4 = (-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$. Set of balls $B_1(a_i)$, $i = \overline{1, 4}$ is admissible, however, the complement of set $B = \bigcup_{i=\overline{1, 4}} B_1(a_i)$ has two connected components.*

Proof

1) Let us show that this set of balls is admissible. Every point x from the tetrahedron $A = a_1a_2a_3a_4$ may be represented in form $x = \alpha_1a_1 + \dots + \alpha_4a_4$, where $\alpha_1 + \dots + \alpha_4 = 1$, $\alpha_i \geq 0, i \in \overline{1, 4}$. Using equation $\alpha_4 = 1 - \alpha_1 - \alpha_2 - \alpha_3$, we are going to construct an inequation which would detect that point $x \in A$ is not contained in ball $B_1(a_4)$:

$$(13) \quad 1 < \|x - a_4\| = \left| \frac{-\alpha_1 + \alpha_2 + \alpha_3}{2} \right| + \left| \frac{\alpha_1 - \alpha_2 + \alpha_3}{2} \right| + \left| \frac{\alpha_1 + \alpha_2 - \alpha_3}{2} \right|.$$

Note that if every expression under the modulus is positive, then the right side of (13) equals $\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) \leq \frac{1}{2}$. So, one of them has to be negative. Without loss of generality, let $\alpha_1 \geq \alpha_2 + \alpha_3$. Then the other two expressions are positive and inequation (13) can be rewritten: $\frac{3\alpha_1 - \alpha_2 - \alpha_3}{2} > 1$. Then $\alpha_1 > \frac{2}{3} + \frac{1}{3}(\alpha_2 + \alpha_3)$.

Using this relation we arrive at:

$$1 - \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3 \geq \frac{2}{3} + \frac{4}{3}(\alpha_2 + \alpha_3).$$

Thus, $\frac{1}{4} - \frac{3}{4}\alpha_4 \geq \alpha_2 + \alpha_3$.

We use the last inequality to estimate the distance between x and the vertex a_1 :

$$\begin{aligned} \|x - a_1\| &= \|\alpha_2(a_2 - a_1) + \alpha_3(a_3 - a_1) + \alpha_4(a_4 - a_1)\| \\ &\leq \alpha_2\|a_2 - a_1\| + \alpha_3\|a_3 - a_1\| + \alpha_4\|a_4 - a_1\| \\ &= 2(\alpha_2 + \alpha_3) + \frac{3}{2}\alpha_4 \leq 2\left(\frac{1}{4} - \frac{3}{4}\alpha_4\right) + \frac{3}{2}\alpha_4 = \frac{1}{2}. \end{aligned}$$

So, we come to the conclusion that the set of balls is admissible.

2) Let $b_1 = (\frac{1}{3}, \frac{1}{12}, \frac{1}{12})$, $b_2 = (\frac{1}{12}, \frac{1}{3}, \frac{1}{12})$, $b_3 = (\frac{1}{12}, \frac{1}{12}, \frac{1}{3})$, $b_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, tetrahedron $\Delta = b_1 b_2 b_3 b_4$.

It is easy enough to show that $\partial\Delta \subset B$, but $\text{int } \Delta \cap B = \emptyset$. ■

5. ABOUT CONTRACTIBILITY OF A PROXIMALLY SMOOTH SETS

Clark, Stern and Wolenski [11] introduced and studied the *proximally smooth sets* in a Hilbert space H . A set $A \subset X$ is said to be proximally smooth with constant R if the distance function $x \rightarrow \rho(x, A)$ is continuously differentiable on set $U(R, A) = \{x \in X : 0 < \rho(x, A) < R\}$. Properties of proximally smooth sets in a Banach space and relations between such sets and akin classes of set, including uniformly prox-regular sets, were investigated in [11]-[12]. We study the sufficient condition of the contractibility for a proximal smooth sets. G.E. Ivanov showed that if $A \subset H$ is proximally smooth (weakly convex in his terminology) with constant R and $A \subset B_r(o)$ with $r < R$, then A is contractible. The following theorem is a generalization of this result.

Theorem 6. *Let X be a uniformly convex and uniformly smooth Banach space. Let A be a closed and proximally smooth with constant R subset of a ball with radius $r < \frac{R}{\zeta_X}$. Then A is contractible.*

Proof.

Note that set $\text{co } A$ is contractible, so a continuous function $F : [0, 1] \times \text{co } A \rightarrow \text{co } A$ such that $F(0, x) = x$, $F(1, x) = x_0$ for all $x \in \text{co } A$ and some $x_0 \in A$ exist. Due to the CHD-constant definition and inequality $r < \frac{R}{\zeta_X}$ the set $\text{co } A$ belongs to the R -neighborhood of the set A . On the other hand, A is proximally smooth and in accordance to paper [13] metric projection mapping $\pi : \text{co } A \rightarrow A$ is single valued and continuous. Finally, we define the mapping $\tilde{F} : [0, 1] \times A \rightarrow A$ as follows $\tilde{F}(t, x) = \pi(F(t, x))$ for all $t \in [0, 1]$, $x \in A$. The mapping \tilde{F} contracts set A to point x_0 . ■

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